

# Dirac Field as a Source of the Inflation in $2 + 1$ Dimensional Teleparallel Gravity

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**Abstract.** In this paper, we study early-time inflation and late-time acceleration of the universe by non-minimally coupling the Dirac field with torsion in the spatially flat Friedman-Robertson-Walker (FRW) cosmological model background. The results obtained by the Noether symmetry approach with and without a gauge term are compared. Additionally, we compare these results with that of the 3+1 dimensional teleparallel gravity under Noether symmetry approach. And we see that the study explains early-time inflation and late-time acceleration of the universe.

*Keywords:*  $2 + 1$  Teleparallel gravity, Dirac field, Noether symmetry, gauge term, inflation,

## 1. Introduction

The recent cosmological observational data point out that our universe has two acceleration periods: early-time and late-time cosmic acceleration. Some cosmological difficulties, such as the initial singularity problem, the flatness problem, and the horizon problem, are resolved by using the early-time inflation scenario with a scalar field as a source of quantum fluctuations [1, 2]. On the other hand, the late-time inflation scenario is due to the observed accelerating expansion of the present universe [3, 4, 5], where the acceleration is explained with a negative pressure component which is called dark energy. The models including different dark energy forms, such as cosmological constant [6, 7], canonical scalar field (i.e. quintessence) [8, 9], and a phantom field [10, 11], have been discussed widely in the context of the standard Einstein relativity.

The Teleparallel gravity is one of the modified theories which has an alternative geometrical formulation of the Einstein's general relativity. It was firstly introduced by Einstein to merge the gravitation with electromagnetism [12]. Though the spacetime structure in the standart Einstein general relativity is described by a Levi-Civita connection which has a free-torsion, but, non-vanishing curvature, the spacetime structure in the teleparallel gravity is constructed by a Weitzenbock connection which is characterized by the metricity condition and vanishing curvature, but it has a torsion [13]. Also, differently from the standard Einstein relativity, the teleparallel gravity is a gauge theory of the spacetime translation group [14, 15, 16, 17].

In recent decades, the 2+1 dimensional gravitation theories, such as the standard Einstein general relativity [18, 19], the new massive gravity [20, 21, 22], the topological massive gravity [23, 24, 25, 26] and the teleparallel gravity [27, 28, 29], have attracted considerable attention in the literature. In particular, the 2+1 dimensional teleparallel theory covers an important place among the 2+1 dimensional gravity theories because it has a Newtonian limit, a black hole solution [27] and a gravitational wave solution named Siklos waves that are a special class of the exact gravitational waves propagating on the AdS background [30]. Also, the other black hole solutions are performed in the context of the 2+1 dimensional teleparallel gravity [31],  $f(T)$  gravity [32], Maxwell- $f(T)$  gravity [33] and torsion scalar coupled with a scalar field non-minimally [34].

As the Dirac theory has a vacuum including Zitterbewegung oscillations between positive and negative energy states and perfectly explains an interaction of a Dirac particle with an atomic structure of the materials, it gives a reasonable cosmological solutions for the early-time inflation and late-time acceleration of the universe [35]. Therefore, in this study, we non-minimally couple the Dirac field with the torsion scalar and consider the Dirac field as a source of early-time inflation and late-time acceleration in the context of the 2+1 dimensional teleparallel gravity on the Friedmann-Robertson-Walker (FRW) background by using the Noether gauge symmetry approach. At same time, we show that the cosmological solutions obtained in the context of the 2+1 dimensional teleparallel gravity are compatible with the results which are in the coupling the Dirac field with the 3+1 dimensional gravity theories [36, 37, 38] and the

2+1 dimensional Einstein gravity [35].

This paper is organized as follows. In the following section, we find the field equations in which the Dirac field is non-minimal coupled with the 2+1 dimensional teleparallel gravity. In Section 3, we search the Noether symmetry for the Lagrangian which is coupled Dirac field with the 2+1 dimensional teleparallel gravity. In Section 4, we obtain solutions of the field equations by using Noether symmetry approach. Furthermore, we find solutions of the field equations under the zero-gauge term condition, and compare these results with the results of study performed in the 3+1 dimensional teleparallel gravity. Finally, in the Section 5, we conclude with a summary of the obtained results. Throughout the paper, we use  $c = G = \hbar = 1$ .

## 2. The Field Equations

The investigations on the interaction between a torsion and a Dirac spinorial field have attracted attention for a long time [39, 40, 41, 42, 43, 44, 45, 46, 47, 48]. The main purpose of these studies are to construct a modified gravitational theories that include a spin effect of the matter fields naturally. Therefore, to consider the inflationary problem of the universe in the framework of the 2+1 dimensional teleparallel gravity non-minimally coupled with Dirac field, we start by writing the action non-minimally coupled a Dirac field with a torsion scalar as follows;

$$\mathcal{A} = \int d^3x |e| \left\{ F(\Psi)T + \frac{i}{2} [\bar{\psi}\sigma^\mu(x)(D_\mu\psi) - (D_\mu\bar{\psi})\sigma^\mu(x)\psi] - V(\Psi) \right\}, \quad (1)$$

where the  $F(\Psi)$  and  $V(\Psi)$  are the coupling functions and the self-interaction potential, respectively. These scalar functions depend on only function of a bilinear Dirac field  $\Psi = \bar{\psi}\psi$ . And the bilinear Dirac field  $\Psi$  is a relativistic invariant quantity [49], where  $\bar{\psi}$  is the adjoint of the Dirac field  $\psi$ , which  $\bar{\psi} = \psi^\dagger \sigma^3$ . In this action,  $|e| = \det(e_\mu^{(i)}(x)) = \sqrt{|g|}$ , where  $e_\mu^{(i)}(x)$  are dreibeins and  $g$  is determinant of the metric tensor,  $g_{\mu\nu}$ , defined in terms of the dreibeins and the 2+1 dimensional Minkowski spacetime metric,  $\eta_{(i)(j)}$ , with the signature  $(1, -1, -1)$ , as  $g_{\mu\nu} = e_\mu^{(i)} e_\nu^{(j)} \eta_{(i)(j)}$ , where  $i$  and  $j$  are the internal (local) spacetime indices, and  $\mu$  and  $\nu$  are the external (global) spacetime indices [50, 51].  $\sigma^\mu(x)$  are the Dirac matrices dependent on spacetime coordinates, and they are defined in terms of the flat spacetime Dirac matrices,  $\bar{\sigma}^i$ , and the dreibeins as follows;

$$\sigma^\mu(x) = e_{(i)}^\mu(x) \bar{\sigma}^i, \quad (2)$$

where  $\bar{\sigma}^i$  are

$$\bar{\sigma}^0 = \sigma^3, \quad \bar{\sigma}^1 = i\sigma^1, \quad \bar{\sigma}^2 = i\sigma^2 \quad (3)$$

and,  $i$  is the imaginary unit (i.e.  $i^2 = -1$ ),  $\sigma^1$ ,  $\sigma^2$  and  $\sigma^3$  are the standard  $2 \times 2$  Pauli matrices [50, 52].  $T$  is the Weitzenbock invariant (i.e. torsion scalar) defined as [32, 33];

$$T = S_\rho^{\mu\nu} T_{\mu\nu}^\rho, \quad (4)$$

where  $T_{\mu\nu}^\rho$  is the Weitzenbock torsion tensor defined in terms of the Weitzenbock connection [13, 32],  ${}^W\Gamma_{\mu\nu}^\rho = e_{(i)}^\rho \partial_\nu e_{\mu}^{(i)}$ ,

$$T_{\mu\nu}^\rho = e_{(i)}^\rho \left[ \partial_\mu e_{\nu}^{(i)} - \partial_\nu e_{\mu}^{(i)} \right], \quad (5)$$

and  $S_\rho^{\mu\nu}$  is a skew-symmetric tensor defined as

$$S_\rho^{\mu\nu} = \frac{1}{4} [T_{\rho}^{\nu\mu} - T_{\rho}^{\mu\nu} + T_{\rho}^{\mu\nu}] + \frac{1}{2} [\delta_\rho^\mu T_{\beta}^{\beta\nu} - \delta_\rho^\nu T_{\beta}^{\beta\mu}]. \quad (6)$$

Furthermore, the torsion tensor can be decomposed into three irreducible parts in the 2+1 dimensional spacetime under the  $SO(2, 1)$  group [53, 54]: the trace part,  $t_\mu = T_{\mu\nu}^\nu$ , the totally antisymmetric part (i.e. pseudo-scalar),  $\chi = \frac{1}{6} \varepsilon^{\mu\nu\lambda} T_{\mu\nu\lambda}$ , and the traceless symmetric tensor,  $X_{\mu\nu}$ . Hence, the torsion tensor is defined in terms of these components as follows:

$$T_{\mu\nu\lambda} = \chi \varepsilon_{\mu\nu\lambda} + \frac{1}{2} (\eta_{\nu\lambda} t_\mu - \eta_{\mu\lambda} t_\nu) + \varepsilon_{\mu\nu\sigma} X_\lambda^\sigma \quad (7)$$

where  $\varepsilon_{\mu\nu\sigma}$  is the three dimensional Levi-Civita symbol ( $\varepsilon_{012}=1$ ). Accordingly, in a 2+1 dimensional spacetime, the minimal coupling between torsion and Dirac field is described by the totally antisymmetric part of the torsion tensor which is a pseudo-scalar [53, 54]. On the other hand, the Weitzenbock connection can be decomposed as  ${}^W\Gamma_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho + K_{\mu\nu}^\rho$ , where  $\Gamma_{\mu\nu}^\rho$  is the Christoffel connection and  $K_{\mu\nu}^\rho$  is the contorsion tensor,  $K_{(j)\mu}^{(i)} = \frac{1}{2} e_{\beta}^{(i)} e_{(j)}^\nu (T_{\nu}^{\beta\mu} + T_{\mu}^{\beta\nu} - T_{\mu\nu}^\beta)$  [43]. Also, the relationship between the Weitzenbock spin connection,  $\omega_{(j)\mu}^{(i)}$ , and general relativity (Lorentz) spin connection,  ${}^0\omega_{(j)\mu}^{(i)}$ , becomes as  $\omega_{(j)\mu}^{(i)} = {}^0\omega_{(j)\mu}^{(i)} + K_{(j)\mu}^{(i)}$ . As the teleparallel gravity is characterized by the vanishing Weitzenbock spin connection (i.e.  $\omega_{(j)\mu}^{(i)} = {}^0\omega_{(j)\mu}^{(i)} + K_{(j)\mu}^{(i)} = 0$ ) [43, 44, 48], the covariant derivative of the Dirac spinor,  $D_\mu \psi$ , and its adjoint,  $D_\mu \bar{\psi}$ , can be expressed respectively as [48, 53]

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi - \frac{i}{4} ({}^0\omega_{(j)\mu}^{(i)}) s_{(i)}^{(j)} \psi, \\ D_\mu \bar{\psi} &= \partial_\mu \bar{\psi} + \frac{i}{4} ({}^0\omega_{(j)\mu}^{(i)}) s_{(i)}^{(j)} \bar{\psi}, \end{aligned} \quad (8)$$

where the  $s_{(i)}^{(j)}$  is a spin operator and it is defined in terms of Dirac matrices:  $s_{(i)}^{(j)} = \frac{i}{2} [\bar{\sigma}_{(i)}, \bar{\sigma}_{(j)}]$ . Furthermore, under these considerations, the minimal coupling the Dirac spinor to the torsion become equivalent to the that of curvature. Accordingly, the Dirac spinor can be consistently coupled to torsion as indicated in previously works [48, 55, 56, 57].

The variation of the action (1) with respect to  $\bar{\psi}$  and  $\psi$  gives us the Dirac equation and its adjoint as follows:

$$i\sigma^\mu(x) D_\mu \psi + [F'T - V'] \psi = 0 \quad (9)$$

$$i(D_\mu \bar{\psi}) \sigma^\mu(x) - [F'T - V'] \bar{\psi} = 0 \quad (10)$$

respectively, where the prime denotes the derivative with respect to the bilinear  $\Psi$ . Additionally, the variation of the action with respect to the dreibein fields,  $e_{\sigma}^{(i)}(x)$ , yields the equation of motion as follows;

$$F \left[ T e_{(i)}^\sigma - 4 T_{\nu(i)}^\alpha S_\alpha^{\nu\sigma} + 4 e^{-1} \partial_\alpha (e S_{(i)}^{\alpha\sigma}) \right] + 4 F' (\partial_\alpha \Psi) S_{(i)}^{\alpha\sigma} = T_{(i)}^\sigma, \quad (11)$$

where  $T_{\nu(i)}^\alpha = e_{(i)}^\lambda T_{\nu\lambda}^\alpha$ ,  $S_{(i)}^{\alpha\sigma} = e_{(i)}^\lambda S_\lambda^{\alpha\sigma}$  and the  $T_{(i)}^\sigma$  is the modified energy-momentum tensor of the Dirac field and its explicit form is expressed as

$$T_{\beta\mu} = \frac{i}{2} [\bar{\psi}\sigma_\beta(x)D_\mu\psi - (D_\mu\bar{\psi})\sigma_\beta(x)\psi] + g_{\beta\mu}[(F'T - V')\Psi + V]. \quad (12)$$

To investigate the inflationary problem in the context of the 2+1 dimensional teleparallel gravity, we will consider the 2+1 dimensional FRW spacetime background that is spatially flat, homogeneous and isotropic universe,

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2], \quad (13)$$

where  $a(t)$  is the scale factor of the universe. For the metric, the dreibein fields and their dual are expressed as  $e_\mu^{(i)} = \text{diag}(1, a(t), a(t))$  and  $e_{(i)}^\mu = \text{diag}(1, 1/a(t), 1/a(t))$ , respectively. Accordingly, the non-null components of the torsion tensor (5) and the  $S_\rho^{\mu\nu}$  skew-symmetric tensor (6) are given by, respectively,

$$T_{01}^1 = T_{02}^2 = H, \quad S_1^{10} = S_2^{20} = \frac{H}{2}, \quad (14)$$

where  $H = \frac{\dot{a}}{a}$  is the Hubble parameter and the dot represents differentiation with respect to cosmic time,  $t$ . Moreover, using the equation (4), the torsion scalar becomes  $T = -2H^2$ . Hence, the Dirac equation (9) and its adjoint (10) are reduced as, respectively,

$$\dot{\psi} + H\psi + iV'\sigma^3\psi + 2iH^2F'\sigma^3\psi = 0, \quad (15)$$

$$\dot{\bar{\psi}} + H\bar{\psi} - iV'\bar{\psi}\sigma^3 - 2iH^2F'\bar{\psi}\sigma^3 = 0. \quad (16)$$

Furthermore, the "00" and "11" components of the equation of motion (11) give us the following:

$$H^2 = \frac{V}{2F} \quad (17)$$

and

$$\frac{\ddot{a}}{a} = -\frac{2HF'\dot{\Psi} + [2H^2F' + V']\Psi - V}{2F} \quad (18)$$

respectively. Also, the "22" component is equivalent to the "11" component.

In order to solve the field equations (17) and (18), a suitable forms of the unknown functions  $F(\Psi)$  and  $V(\Psi)$  must be given. However, in this study, we calculate the forms of these functions by using the Noether gauge symmetry approach method. For this reason, firstly, we must construct the Lagrangian of the system defined by equation (1). Therefore, with similar calculations in the [35], the point-like Lagrangian of the system can be written from the action (1) in the following form:

$$L = 2F\dot{a}^2 - \frac{ia^2}{2} (\bar{\psi}\sigma^3\dot{\psi} - \dot{\bar{\psi}}\sigma^3\psi) + a^2V. \quad (19)$$

The Dirac's equations for the spinor field  $\psi$  and its adjoint  $\bar{\psi}$  are obtained from the Lagrangian (19) such that the Euler-Lagrange equations of the  $\psi$  and  $\bar{\psi}$  are, respectively,

$$\dot{\psi} + H\psi + iV'\sigma^3\psi + 2iH^2F'\sigma^3\psi = 0, \quad (20)$$

$$\dot{\bar{\psi}} + H\bar{\psi} - iV'\bar{\psi}\sigma^3 - 2iH^2F'\bar{\psi}\sigma^3 = 0, \quad (21)$$

Furthermore, letting the point-like Lagrangian (19) and Dirac's equations, we find the second order Euler-Lagrange equation for  $a$ , i.e. the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{p_f}{2F}, \quad (22)$$

where  $p_f$ ,

$$p_f = 2HF'\dot{\Psi} + [2H^2F' + V']\Psi - V, \quad (23)$$

is the pressure of the Dirac field. Furthermore, the constraint equation for the energy function ( $E_L = 0$ ) associated with the point-like Lagrangian (19) is written as

$$E_L = \frac{\partial L}{\partial \dot{a}}\dot{a} + \frac{\partial L}{\partial \dot{\psi}}\dot{\psi} + \dot{\bar{\psi}}\frac{\partial L}{\partial \dot{\bar{\psi}}} - L, \quad (24)$$

[58, 59], which it is equivalent to the Friedmann equation:

$$H^2 = \frac{\rho_f}{2F}, \quad (25)$$

where  $\rho_f$  represents the effective energy density and it is given by the self-interaction potential of a Dirac field in the following form;

$$\rho_f = V. \quad (26)$$

As can be seen, the Dirac equation (15) and its adjoint (16) as well as the equations of motion (17) and (18) were recovered again by using the Lagrangian formalism.

### 3. The Noether symmetry approach

The symmetry conception has always played a central role in physics because it is directly associated with the conservation laws of a dynamical system [60]. These connections between symmetries and conservation laws of a dynamical system is expressed by the Noether's theorem [61]. So, in this study, the existence of Noether symmetries lead to a specific form of coupling function and the self-interaction potential, which they are important for obtaining the exact solutions of the fields equations [58, 59, 62, 63, 64, 65, 66]. Mathematically, the Noether symmetry condition for the system where the Dirac field coupled with the 2+1 dimensional gravity, with a gauge term,  $B$ , can express as follows [35, 67, 68, 63, 69];

$$\mathbf{X}^{[1]}L + L(D_t\tau) = D_tB, \quad (27)$$

where  $B=B(t, a, \psi_j, \psi_j^\dagger, \dot{a}, \dot{\psi}_j, \dot{\psi}_j^\dagger)$  is a gauge term,  $D_t$  is the operator of the total differentiation with respect to  $t$

$$D_t = \frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a} + \sum_{j=1}^2 \left( \dot{\psi}_j \frac{\partial}{\partial \psi_j} + \dot{\psi}_j^\dagger \frac{\partial}{\partial \psi_j^\dagger} \right), \quad (28)$$

and  $\mathbf{X}^{[1]}$ , where

$$\begin{aligned} \mathbf{X}^{[1]} = & \mathbf{X} + (D_t\alpha - \dot{a}D_t\tau) \frac{\partial}{\partial \dot{a}} + \sum_{j=1}^2 \left[ (D_t\beta_j - \dot{\psi}_j D_t\tau) \frac{\partial}{\partial \dot{\psi}_j} \right] \\ & + \sum_{j=1}^2 \left[ (D_t\gamma_j - \dot{\psi}_j^\dagger D_t\tau) \frac{\partial}{\partial \dot{\psi}_j^\dagger} \right], \end{aligned} \quad (29)$$

is the first-order prolongation of the vector field  $\mathbf{X}$  given by

$$\mathbf{X} = \tau \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \sum_{j=1}^2 \left( \beta_j \frac{\partial}{\partial \psi_j} + \gamma_j \frac{\partial}{\partial \psi_j^\dagger} \right), \quad (30)$$

where the coefficients  $\tau$ ,  $\alpha$ ,  $\beta_j$  and  $\gamma_j$  are dependent on the variables  $t, a, \psi_j, \psi_j^\dagger, \dot{a}, \dot{\psi}_j, \dot{\psi}_j^\dagger$ . Letting the spinor field  $\psi = (\psi_1, \psi_2)^T$  and its adjoint,  $\bar{\psi} = \psi^\dagger \sigma^3$ , the point-like Lagrangian (19) is reduced to the following form:

$$L = 2F\dot{a}^2 - \frac{ia^2}{2} \left[ \sum_j^2 (\psi_j^\dagger \dot{\psi}_j - \dot{\psi}_j^\dagger \psi_j) \right] + a^2 V. \quad (31)$$

Hence, the point-like Lagrangian (31) and the Noether gauge symmetry condition (27) leads to the following partial differential equations obtained from the fact that the coefficients of the  $\dot{a}^3, \dot{a}^2, \dot{a}, \dot{\psi}_j, \dot{\psi}_j^\dagger, \dot{a}\dot{\psi}_j, \dot{a}\dot{\psi}_j^\dagger, \dot{a}^2\dot{\psi}_j$ , and  $\dot{a}\dot{\psi}_j^\dagger$  vanish, separately:

$$\begin{aligned} 2F \left( 2 \frac{\partial \alpha}{\partial a} - \frac{\partial \tau}{\partial t} \right) + 2F' \sum_{j=1}^2 \epsilon_j (\beta_j \psi_j^\dagger + \gamma_j \psi_j) &= 0, \\ 4F \frac{\partial \alpha}{\partial \psi_j} &= 0, \quad 4F \frac{\partial \alpha}{\partial \psi_j^\dagger} = 0, \\ 2F \frac{\partial \tau}{\partial \psi_j} &= 0, \quad 2F \frac{\partial \tau}{\partial \psi_j^\dagger} = 0, \quad 2F \frac{\partial \tau}{\partial a} = 0, \\ i\alpha \psi_j + \frac{ia}{2} \beta_j - \frac{ia}{2} \sum_{i=1}^2 \left( \frac{\partial \beta_i}{\partial \psi_j^\dagger} \psi_i^\dagger - \frac{\partial \gamma_i}{\partial \psi_j^\dagger} \psi_i \right) + aV \frac{\partial \tau}{\partial \psi_j^\dagger} - \frac{1}{a} \frac{\partial B}{\partial \psi_j^\dagger} &= 0, \\ i\alpha \psi_j^\dagger + \frac{ia}{2} \gamma_j + \frac{ia}{2} \sum_{i=1}^2 \left( \frac{\partial \beta_i}{\partial \psi_j} \psi_i^\dagger - \frac{\partial \gamma_i}{\partial \psi_j} \psi_i \right) - aV \frac{\partial \tau}{\partial \psi_j} + \frac{1}{a} \frac{\partial B}{\partial \psi_j} &= 0, \\ 4F \frac{\partial \alpha}{\partial t} - \frac{ia^2}{2} \sum_{j=1}^2 \left( \frac{\partial \beta_j}{\partial a} \psi_j^\dagger - \frac{\partial \gamma_j}{\partial a} \psi_j \right) + a^2 V \frac{\partial \tau}{\partial a} - \frac{\partial B}{\partial a} &= 0 \end{aligned} \quad (32)$$

and also, the remaining expressions satisfy the following equation:

$$\begin{aligned} (2\alpha + a \frac{\partial \tau}{\partial t})V - \frac{1}{a} \frac{\partial B}{\partial t} + aV' \sum_{j=1}^2 \epsilon_j (\beta_j \psi_j^\dagger + \gamma_j \psi_j) \\ - \frac{ia}{2} \sum_{j=1}^2 \left( \psi_j^\dagger \frac{\partial \beta_j}{\partial t} - \psi_j \frac{\partial \gamma_j}{\partial t} \right) &= 0, \end{aligned} \quad (33)$$

where

$$\epsilon_j = \begin{cases} 1 & \text{for } j = 1 \\ -1 & \text{for } j = 2. \end{cases} \quad (34)$$

Then, the complete solutions of the (32) are obtained as follows:

$$\begin{aligned} \alpha &= -\frac{c_1(k+1)}{2(k-1)}a, \quad \beta_j = \frac{c_1(k+1)}{2(k-1)}\psi_j + \epsilon_j \beta_0 \psi_j, \\ \gamma_j &= \frac{c_1(k+1)}{2(k-1)}\psi_j^\dagger - \epsilon_j \beta_0 \psi_j^\dagger, \quad \tau = c_1 t + c_2, \quad B = c_4 \end{aligned} \quad (35)$$

and the coupling function,  $F(\Psi)$ , is found as

$$F(\Psi) = f_0 \Psi^{\frac{2k}{k+1}}, \quad (36)$$

where the  $c_1, c_2, c_4, f_0$  and  $k$  ( $k \neq 1$ ) are integration constants. Finally, inserting (35) and (36) in (33), the self-interaction potential is determined as

$$V(\Psi) = \lambda \Psi^{\frac{2}{k+1}}, \quad (37)$$

where  $\lambda$  is integration constants.

Using (35), the corresponding Noether gauge symmetry generators are obtained as,

$$\begin{aligned} \mathbf{X}_0 &= \frac{\partial}{\partial t}, \\ \mathbf{X}_1 &= t \frac{\partial}{\partial t} - \frac{k+1}{2(k-1)} \left[ a \frac{\partial}{\partial a} - \sum_{i=1}^2 \left( \psi_i \frac{\partial}{\partial \psi_i} + \psi_i^\dagger \frac{\partial}{\partial \psi_i^\dagger} \right) \right], \\ \mathbf{X}_2 &= \sum_{i=1}^2 \epsilon_i \left( \psi_i \frac{\partial}{\partial \psi_i} - \psi_i^\dagger \frac{\partial}{\partial \psi_i^\dagger} \right). \end{aligned} \quad (38)$$

Furthermore, these generators satisfy the following commutation relations;

$$[\mathbf{X}_0, \mathbf{X}_1] = \mathbf{X}_0, \quad [\mathbf{X}_0, \mathbf{X}_2] = [\mathbf{X}_1, \mathbf{X}_2] = 0 \quad (39)$$

Thanks to the Noether theorem, if the vector field  $\mathbf{X}$  is a Noether gauge symmetry corresponding to the Lagrangian  $L$ , then, the following equivalent is a first integral (i.e. conserved quantities of the system) associated with  $\mathbf{X}$ :

$$\begin{aligned} I &= \tau L + (\alpha - \tau \dot{a}) \frac{\partial L}{\partial \dot{a}} + \sum_{j=1}^2 \left[ (\beta_j - \tau \dot{\psi}_j) \frac{\partial L}{\partial \dot{\psi}_j} \right] \\ &+ \sum_{j=1}^2 \left[ (\gamma_j - \tau \dot{\psi}_j^\dagger) \frac{\partial L}{\partial \dot{\psi}_j^\dagger} \right] - B. \end{aligned} \quad (40)$$

Moreover, there are three first integrals (or conserved quantities) associated with the Noether gauge symmetries:

$$I_0 = -2F\dot{a}^2 + a^2V, \quad (41)$$

$$I_1 = tI_0 - \frac{2F(k+1)}{(k-1)}a\dot{a}, \quad (42)$$

$$I_2 = -\frac{ia^2\Psi}{2}. \quad (43)$$

It is important to note that the first integral (41) is associated to the energy function (24), since the first integral  $I_0$  vanishes identically.

#### 4. The solutions of the field equations

To determine the time evolution of the scale factor,  $a(t)$ , firstly, we need to know the form of bilinear function,  $\Psi$ . Thus, we insert the coupling function  $F$  given by (36) in the Dirac equation (20) and in its adjoint (21) we gets

$$\dot{\Psi} + 2\frac{\dot{a}}{a}\Psi = 0. \quad (44)$$



Then, the form of the bilinear function is calculated as

$$\Psi = \frac{\Psi_0}{a^2}, \quad (45)$$

where  $\Psi_0$  is a constant of integration. Then, the first integral equation (43) is found  $I_2 = -\frac{\Psi_0}{2}$ . Furthermore, inserting the equations (45) and the coupling function (36) in the first integral (42), we get

$$\dot{a} + K a^{\frac{3k-1}{k+1}} = 0, \quad (46)$$

where

$$K = \frac{(k-1)I_1}{2(k+1)f_0\Psi_0^{\frac{2k}{k+1}}}, \quad (47)$$

and then integrating this equation, the scale factor  $a(t)$  finds as

$$a(t) = \left[ \frac{2K(k-1)}{k+1}t + a_0 \right]^{\frac{k+1}{2(1-k)}}, \quad (48)$$

where  $a_0$  is an integration constant. Using the solution (48) in the acceleration equation (22) and in the Friedmann equation (25), we obtain a constraint relations between the constants as  $\lambda = I_1^2(k-1)^2/2\Psi_0^2 f_0(k+1)^2$ . Under these conditions, the scale factor that is obtained from the Noether gauge symmetry represents a power law expansion for the universe. Moreover, using the definition of the deceleration parameter  $q$ , namely  $q = -a\ddot{a}/\dot{a}^2$ , it can be seen that the evolution of the universe has undergone three different processes. Using (48), the deceleration parameter calculated as  $q = (1-3k)/(k+1)$ . From the deceleration parameter and (48), we see that the scale factor,  $a(t)$ , is up to the parameter  $k$ , and it corresponds to these different cosmological models for  $k$  values: firstly, for  $k \in (-\infty, -1) \cup (1/3, \infty)$  the deceleration parameter becomes  $q < 0$ , and hence obey an accelerated power law expansion; secondly, for  $k \in (-1, 1/3)$ , the deceleration parameter becomes  $q > 0$ , and therefore a decelerated expansion occurs; thirdly, for the  $k=1/3$  the deceleration parameter becomes  $q=0$ , and this case corresponds to a uniformly (i.e. monotonically) expanding universe model. Also, for  $k=1/3$ , we have

$$a(t) = \frac{I_1}{4f_0\sqrt{\Psi_0}}t + a_0. \quad (49)$$

This solution, corresponding to a universe with the pressureless Dirac field, is similar to the matter (or dust) dominant universe in the standard  $2+1$  dimensional general relativity [70]. Therefore, this solution shows that the Dirac field behaves as a standard pressureless matter field in the  $2+1$  dimensional teleparallel gravity.

Furthermore, we can define the equation of state parameter of the fermionic field by using the energy density (26) and pressure (23) as  $\omega_f = p_f/\rho_f$  to search that whether the fermionic field can provide alternative for dark energy or not. Given the equations (36), (37), (45) and (48), we obtain

$$\omega_f = \frac{1-3k}{k+1}. \quad (50)$$

Based on the recently astrophysical observational data [71, 72], the equation of the state parameter tends to value  $-1$ . Additionally, as the equation of the state parameter is less than  $-1$  the dark energy is described by phantom, but, for  $\omega \in (-1, -1/3)$ , the quintessence dark energy is observed and the case  $\omega = -1$  corresponds to the cosmological constant. In the present work, we observed that, the case  $k \in (1/2, 1)$  corresponding to the quintessence phase, and the case  $k \in (-\infty, -1) \cup (1, \infty)$  is corresponding to the phantom phase. In both cases, the universe is both expanding and accelerating. Therefore, the results show that the fermionic field may behave like both the quintessence and the phantom dark energy field in the late-time universe.

For the case  $k=1$ , the coupling and potential functions have linear forms of  $\Psi$  from the equation (36) and (37) as follows:

$$\begin{aligned} F(\Psi) &= f_0 \Psi, \\ V(\Psi) &= \lambda \Psi. \end{aligned} \tag{51}$$

Therefore, without the Noether symmetry, the Friedmann equation (25) is reduced to

$$\frac{\dot{a}}{a} - \sqrt{\frac{\lambda}{2f_0}} = 0, \tag{52}$$

which has the following solution

$$a(t) = a_0 e^{H_0 t}, \quad \text{where} \quad H_0 = \sqrt{\frac{\lambda}{2f_0}}, \tag{53}$$

where  $a_0$  is a constant. It is clear that this solution is corresponding to the de Sitter solution that describes an inflationary epoch of the universe. Hence, it can be concluded that the Dirac field plays a role as an inflaton field in the  $2+1$  dimensional teleparallel gravity. In this situation, from Eqs. (26) and (23), the energy density and the pressure of the Dirac field are given by

$$\rho_f = \frac{\lambda \Psi_0}{a_0^2} e^{-2H_0 t}, \quad p_f = -\rho_f. \tag{54}$$

Furthermore, the state parameter calculated as  $\omega_f = -1$  corresponds to the cosmological constant.

It is interesting to note that in the case of vanishing the gauge term (i.e.,  $B=0$ ), the partial differential equations Eqs.(32)-(33) are reduced to the equations for the usual Noether symmetry approach. In this case, in a similar form of the Eqs.(48), the scale factor obtained as

$$a(t) = [\tilde{a}_0 (k-1)(t-a_1)]^{\frac{1}{1-k}} \tag{55}$$

where  $a_1$  integration constant and  $\tilde{a}_0 = \frac{\lambda \Psi_0^{1-k}}{2f_0}$ . This solution is the same as the results obtained in the context of non-minimal coupling the fermionic field to the torsion in  $3+1$  dimensional teleparallel gravity [38]. According to this solution, the deceleration parameter becomes  $q = -k$ , and hence, the universe is accelerating for  $k > 0$ , decelerating for  $k < 0$  and the case  $k < 0$  is corresponds to uniformly expanding universe model. Also, the equation of state parameter,  $\omega$ , calculated as  $\omega = -k$ , so that, the cases  $k \in (1/3, 1)$ ,

$k \in (1, \infty)$  and  $k=1$  are correspond to the quintessence phase, to the phantom phase and to the cosmological constant, respectively. Hence, we can say that the fermionic field causes the same physical effect on the universe's history in both 2+1 and 3+1 dimensional teleparallel gravity theories under the Noether symmetry condition. On the other hand, there is a similar equivalence between the 2+1 [35] and 3+1 [37] dimensional Einstein gravities under Noether symmetry approach in the presence of the non-minimal coupling the fermionic field to curvature scalar.

On the other hand, the minimal coupling case is characterized by  $F'=0$ , i.e. by constant coupling function  $F=n$ . If we choose  $n=1/2$  and ignore the gauge term (i.e.,  $B=0$ ), the solutions of the partial differential equations (32) are

$$\begin{aligned}\alpha &= \alpha_0, \quad \beta_j = (\epsilon_j \beta_0 - \frac{\alpha_0}{a}) \psi_j, \\ \gamma_j &= (-\epsilon_j \beta_0 - \frac{\alpha_0}{a}) \psi_j^\dagger\end{aligned}\tag{56}$$

and from the Eqs.(33) the self-interaction potential is determined as

$$V(\Psi) = V_0 \Psi.\tag{57}$$

Hence, from the (22) and (25), the scale factor is calculated as,

$$a(t) = \sqrt{V_0 \Psi_0} t + a_0.\tag{58}$$

According to (58), the deceleration parameter becomes  $q = 0$ , Hence, we can say that, in the minimal coupling case, the universe in the 2+1 dimensional teleparallel gravity in the presence of the Dirac field is expanding uniformly, only. Furthermore, this solution describes a universe dominated by pressureless matter (or dust) [70]. Also, this solution is equivalent to the solution of the 2+1 Einstein general relativity in the case of the minimal coupling of fermionic fields with curvature scalar [35]. On the other hand, in the non-minimal coupling case, 2+1 dimensional teleparallel gravity has a similar physical results about the history of the universe that of 3+1 dimensional teleparallel gravity [38], whereas it is different from the 3+1 dimensional standard Einstein theory, because, the 3+1 dimensional standard Einstein theory explains the early-time inflation period, only [37], but, as shown in this study, 2+1 dimensional teleparallel gravity explains both the early-time and late-time inflation periods.

## 5. Concluding remarks

The investigations to obtain a suitable components that can explain the early-time inflation and late-time acceleration epoch of the universe is one of the main topics of modern cosmology. In this context, we consider the non-minimal coupling of the Dirac field to torsion in the context of teleparallel theory of gravity in 2+1 dimensional Weitzenböck spacetime to investigate whether dirac fields may be responsible for the early-time inflation and late-time acceleration. Therefore, the identification of the non-minimal coupling and the self-interaction functions are performed by using the Noether symmetry approach with and without a gauge term. From this approach, the obtained results can be summarized as follows:

- For the case  $k=1$ , the scale factor obtained in Eqs. (53) is  $a(t)=a_0e^{H_0t}$ , indicate the inflation epoch of the universe. Also this solution corresponds to the vacuum energy which characterized by cosmological constant.
- The case  $k \in (1/2, 1)$  conform to the quintessence dark energy component that is used to explain the late-time acceleration.
- The case  $k \in (-\infty, -1) \cup (1, \infty)$  corresponds to the phantom dark energy component. This component is, also, used to explain the late-time acceleration.
- The cosmological solution for  $k=1/3$  corresponds to a uniformly (i.e. monotonically,) expanding universe model. Also, it is obtained a solution correspond to the matter (or dust) dominant universe with the pressure of the Dirac field  $p_f=0$  as in the  $2+1$  dimensional General Relativity [70]. Hence, it shows that the Dirac field behaves as a standard pressureless matter field in the  $2+1$  dimensional teleparallel gravity.
- Under the usual Noether symmetry condition (i.e.,  $B=0$ ), the scale factor  $a(t)$  is computed as  $a(t)=[\tilde{a}_0(k-1)(t-a_1)]^{\frac{1}{1-k}}$ , where  $a_1$  integration constant and  $\tilde{a}_0=\frac{\lambda\Psi_0^{1-k}}{2f_0}$ . It is interesting to note that this solution is the same as the results obtained in the context of non-minimal coupling the fermionic field to the torsion in  $3+1$  dimensional teleparallel gravity [38]. Hence, we can say that, in the presence of the fermionic field, the  $2+1$  dimensional teleparallel gravity has similar physical results about the evolution of the universe that of the  $3+1$  dimensional teleparallel gravity thanks to the Noether symmetry.
- In the case of the minimal coupling (i.e.,  $F=1/2$ ) and under the usual Noether symmetry condition (i.e.,  $B=0$ ) our solutions are reduced to  $2+1$  dimensional Einstein general relativity solutions, as well.
- It is important to emphasize that for the non-minimal coupling and under the Noether symmetry approach with a gauge term, the  $2+1$  dimensional Einstein gravity [35] has similar physical results about the history of the universe that of both  $2+1$  and  $3+1$  dimensional teleparallel gravities [38] in the presence of the fermionic field.

Finally, it can be said that the Dirac field is an inflaton describing the acceleration of the Universe both in the early time epoch and in the late-time epoch.

## Acknowledgments

The Authors thanks Dr. Yusuf Kucukakca for useful discussion. Also, the authors are grateful to the anonymous referees for their valuable comments. This work was supported by the Scientific Research Projects Unit of Akdeniz University.

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